Thursday, October $11^{th}$, 2007
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# MA 3025, Sample Exam 1, page 1

Solve 4 of the 5 problems (25 points each) or all five of them (20 points each). You may solve all 5 and decide to turn in 4 (please cross off the one you don't want me to grade then).

1 [25/20 Pts] In the questions below suppose $A = \{x, y\}$ and $B = \{x, \{x\}\}$. Mark each statement TRUE or FALSE, no explanations are needed.

(a) $x \subseteq B$.
Ans: False since $x$ is not a set (note that $x \in B$)

(b) $\varnothing \in P(B)$, where $P(B)$ is the power set of $B$.
Ans: True, because the power set of any set has the element empty set in it.

(c) $\{x\} \subseteq A - B$.
Ans: False since $A - B = \{y\}$

(d) $|P(A)| = 4$, $P(A)$ is the power set of A.
Ans: True since $|P(A)| = 2^{|A|} = 4$

2. [25/20 Pts] (a) Find $\bigcup_{i=1}^{+\infty} (i, \infty)$.

Ans: $(1, \infty)$

(b) Find gcd(78, 35)

Ans:
$$\begin{aligned} 78 &= 2 \cdot \underline{35} + \underline{8} \\ \underline{35} &= 4 \cdot \underline{8} + \underline{3} \\ \underline{8} &= 2 \cdot \underline{3} + \underline{2} \\ \underline{3} &= 1 \cdot \underline{2} + \underline{1} \end{aligned}$$

(c) Find the coefficient of $y^6$ in the expansion of $(2x - y)^{11}$.

Ans: $\binom{11}{6} 2^5 (-1)^6 x^5 = 14784x^5$ since we would look at $(2x - y)^{11}$ as a function of y only.

(Note that generally the coefficient of $x^5y^6$ is $\binom{11}{6} 2^5 (-1)^6$ or $\binom{11}{5} 2^5 (-1)^6 = 14,784$.

The number "11 choose 6" is the binomial coefficient from the Pascal's triangle.)

(d) How many bit strings of length 6 begin with 1 or end with 1

Ans: $2^5 + 2^5 - 2^4 = 48$

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3. [25/20 Pts] Prove or disprove: The average of two rational numbers is rational.

Ans: True.

Proof: Let *x* and *y* be the two rational numbers. Then $x=\frac{a}{b}$ and $y=\frac{c}{d}$ with *a, b, c, d* integers and $b\neq 0, d\neq 0$. Then the average of *x* and *y* is $\frac{x+y}{2}=\frac{\frac{a}{b}+\frac{c}{d}}{2}=\frac{ad+bc}{2bd}$. Since $ad + bc$ and $2bd$ are integers, and also $2bd\neq 0$, we have that the average is rational. ■

4. [25/20 Pts] Find all solutions to the system of system of congruences:

$x \equiv 7$ (*mod* 10)
$x \equiv 4$ (*mod* 11)

Solution: We use the Chinese Remainder Thm. with $m = 110$.

| | | |
|---|---|---|
| $a_1 = 7$ | $M_1 = 11$ | $y_1 = 1$ |
| $a_2 = 4$ | $M_2 = 10$ | $y_2 = 10$ |

And so the solution is $x = 7 \cdot 11 \cdot 1 + 4 \cdot 10 \cdot 10$ (*mod* 110) = 37 (*mod* 110), or
x = 110*k* + 37, where *k* is an integer.

5. [25/20 Pts] Use a combinatorial proof to show that $\binom{3n}{3}=\binom{2n}{3}+\binom{n}{3}+n\cdot\binom{2n}{2}+2n\cdot\binom{n}{2}$

Proof: The number on the left hand side is the number of 3-subsets of a 3n-set. For the right hand side, let the 3n-set contain say, 2n red and n blue elements. There are $\binom{2n}{3}\binom{n}{0}$ red 3-subsets, $\binom{2n}{0}\binom{n}{3}$ blue 3-subsets, $\binom{2n}{2}\binom{n}{1}=n\cdot\binom{2n}{2}$ 2 red and 1 blue subsets, and $\binom{2n}{1}\binom{n}{2}$ 1 red and 2 blue subsets. We thus have a total of $\binom{2n}{3}+\binom{n}{3}+n\cdot\binom{2n}{2}+2n\cdot\binom{n}{2}$ subsets. ■